THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH3070 Introduction to Topology 2017-2018
Solution of Tutorial Classwork 4

1. Pick any two distinct points $(x_1, x_2, \dots)$ and $(y_1, y_2, \dots)$. Then there exist $k \in \mathbb{N}$ such that $x_k \neq y_k$. Since X_k is a Hausdorff space, there exists $U_k, V_k \in \mathfrak{T}_{X_k}$ such that $x_k \in U_k$, $y_k \in V_k$ and $U_k \cap V_k = \emptyset$. This implies that

$$(x_1, x_2, \dots, x_k, \dots) \in X_1 \times X_2 \times \dots \times X_{k-1} \times U_k \times X_{k+1} \times \dots \quad ,$$
$$(y_1, y_2, \dots, y_k, \dots) \in X_1 \times X_2 \times \dots \times X_{k-1} \times V_k \times X_{k+1} \times \dots$$

and

$$(X_1 \times X_2 \times \dots \times X_{k-1} \times U_k \times X_{k+1} \times \dots) \cap (X_1 \times X_2 \times \dots \times X_{k-1} \times V_k \times X_{k+1} \times \dots) = \emptyset$$

Hence $(\prod_{n\in\mathbb{N}} X_n, \mathfrak{T}_{\text{prod}})$ is Hausdorff.

2. (a) Pick any open set $\prod_{x\in[0,1]} U_x$ containing the point f. Then there exists $x_1, x_2, \dots x_n$ such that $U_{x_n} \neq \mathbb{R}$ and $U_x = \mathbb{R}$ for any $x \neq x_1, x_2, \dots x_n$. Note that $0 \in U_{x_k}$ for $k = 1, 2, \dots n$. So there exists $\epsilon_k > 0$ such that $0 \in (-\epsilon_k, \epsilon_k) \subset U_k$ for all $k = 1, 2, \dots n$. In particular, there exists $N_k > 0$ such that $\frac{1}{n} < \epsilon_k$ for all $n \geq N_k$. Take $N = \max\{N_1, N_2, \dots, N_k\}$. Then for all $n \geq N$, we have $f_n \in \prod_{x\in[0,1]} U_x$. Hence $f_n \to f$ in product topology.

   (b) * Define $U_0 = \mathbb{R}$ and $U_x = (-x, x)$ for all $x \in (0, 1]$. Since $U_x \in \mathfrak{T}_{\mathbb{R}}$ for all $x \in [0, 1]$, $\prod_{x\in[0,1]} U_x$ is open in box topology (but not in product topology). Note that $f \in \prod_{x\in[0,1]} U_x$. Furthermore, suppose $f_n \to f$. Then there exists $N > 0$ such that $f_n \in \prod_{x\in[0,1]} U_x$ for all $n \geq N$. In particular we have $\frac{1}{N} \in U_x = (-x, x)$ for all $x \in (0, 1]$. Since

$$\frac{1}{N} \notin U_{\frac{1}{2N}} = (-\frac{1}{2N}, \frac{1}{2N}),$$

   contradiction. Hence $f_n \not\to f$ in box topology.